Method of Modeling, Parameter and State Estimation of Nonlinear Systems

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Abstract

THE problem of obtaining the simultaneous weighted least squares (WLS) estimate of the parameters $\boldsymbol{p}$ and states $r(\boldsymbol{p},t)$ of a given general first- or second-order nonlinear dynamic system is addressed, respectively, via the use of a quadratic (QSS) or cubic spline series (CSS) time representation of $r(\boldsymbol{p},t)$. The measured states or functions of the states are assumed to be the true values corrupted by additive zero-mean uncorrelated random noise. The CSS is described in Eq. (6) below.[1]

Unknown state functional dependence in the dynamics, e.g., aerodynamic force modeling of an unknown re-entry vehicle, is effectively represented in the time domain with a vector $\boldsymbol{B}(t)$ of unknown time functions. The $\boldsymbol{B}(t)$ are also parameterized via the spline time series.[1] This has the ability to accurately represent both complex state functional dependence and "well behaved" solutions of differential equations with a relatively modest number of terms and, hence, unknown coefficients. The initial conditions and unknown system parameters are WLS estimated from a given set of discrete noisy measurements on the system output at observation times $t_i\,(i=1,I)$ whence $r(\boldsymbol{p},t)$ and its time derivatives at any t, follow immediately as a linear function of $\boldsymbol{p}$.

A variety of analytical spline algorithms are described that include iterative batch processing for maximum accuracy and sequential recursive processing for real time operation. Sensitivity differential equations, as in standard maximum likelihood parameter estimation, and error covariance propagation equations, as with the extended Kalman filter, are eliminated. Precision, generality, flexibility, and computational efficiency are achieved. These are illustrated in the full paper[2,7] with a numerical example based upon the tracking of Van der Pol's equation. The solution is attained via hand calculation for comparison with familiar methods. Additionally, analytical solution expressions are obtained for a practical naval combat weapon systems problem.

Content

Earlier spline methods having similar characteristics were applied successfully to atmospheric trajectory and ballistic coefficient history estimation from radar observations of numerous re-entry vehicle flight tests.[3] Small zero mean radar coordinate residuals and "practical" congruency of the spline estimated ballistic coefficient (β) history up to about 300,000 ft altitude, with the (β) obtained independently from a high accuracy accelerometer carried on board the actual vehicle, was observed. Simulation studies[3-4] conducted for a variety of radar geometries, re-entry vehicles/conditions, and including the three-degree-of-freedom (3DOF) processing of 6DOF generated radar observations for different angle of attack histories[3] indicated good precision and near optimality (correct mean, variance, statistics). Second-order differential equation constraints (Newton's 2nd Law) on $r(\boldsymbol{p},t)$ were satisfied by direct substitution into the QSS.

Spline time series modeling of aircraft aerodynamic force and moment coefficients at high angles of attack, followed by nonlinear parameter and state estimation in 6DOF with nonlinear measurement/state relationships, 24 measurement sets of data that included 9 aircraft mounted accelerometers and 3 rate gyros, is described in Ref. 5. The noisy aileron, elevator, and rudder "$\delta(t)$" control time histories were smoothed via the QSS with the $\ddot{\delta}$ parameters being treated as independent (no dynamic constraints). The results invariably passed right through the center of the $\delta(t)$ data.

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The standard eight aircraft nonlinear kinematic differential equations of order 1 were differentiated once and substituted into the QSS for the $\ddot{r}_k$ parameters, thus satisfying the 6DOF equations at the centers ξ_k of the spline regions of Fig. 2 in Ref. 1.

A variety of aircraft 6DOF simulation studies conducted for high angles of attack, using wind tunnel tabular aerodynamic force and moment coefficient data together with straight numerical integration of the 6DOF equations as the "truth model," again indicated the near optimality of the spline technique(s). Other applications of the QSS or CSS include radar cross section ship classification[6] and optimal thrust vector controls for conducting atmospheric intercepts.[1]

The CSS with the $\dddot{r}_k$ parameters replaced by the first difference of the $\ddot{r}$ values at the adjacent knot positions, was successfully tested in Ref. 7 as an algorithm for the real time numerical integration of the three-point-mass re-entry trajectory and associated 18 initial condition sensitivity differential equations. A synopsis of the mathematical development of the full paper is presented here for nonlinear differential equation systems of order 2.

Mathematical Problem Formulation and Solution for Second-Order Dynamic Systems

A general second-order nonlinear dynamic system with $r^T \equiv [r_1,\ldots,r_D]$ and $\boldsymbol{B}^T(t) \equiv [B^1,\ldots,B^e\ldots,B^E]$ can be written

$$\ddot{r}(t) = \bar{f}(t,r,\dot{r}) + \sum_{e=1}^{E} \bar{g}^e(t,r,\dot{r})B^e(t) \quad \text{given dynamics} \tag{1}$$

where $r \equiv r(\boldsymbol{p},t)$ and $\boldsymbol{p}$ is the vector of unknown parameters. $\bar{f}$, $\bar{g}^e$ are $D\times 1$ given analytical vector functions of t, r, and $\dot{r}$. The $B^e(t)$ are unknown scalar functions of t that are judicious replacements of terms or whole groups of terms in the original differential equations with uncertain state functional dependence.

When $\boldsymbol{B}(t)$ occurs nonlinearly, linearization about a nominal, $\boldsymbol{B}^0(t)$, produces the form Eq. (1) which is therefore not restrictive. A * superscript above a quantity is used to denote a measured value (with noise) and a λ overscript the λth time derivative of that quantity, where $\lambda=0$ denotes the quantity itself. In general, a given nonlinear vector function, $\boldsymbol{H}$, of the states and $\boldsymbol{B}(t)$ is measured at discrete observation times t_i, $(i=1,I)$. Letting $\lambda=0,1,2,$

$$\overset{\lambda}{r}{}_i^* = \overset{\lambda}{\boldsymbol{H}}(t_i, r_i,\ldots,\overset{\lambda}{r}_i,\ \boldsymbol{B}_i,\ldots,\overset{\lambda}{\boldsymbol{B}}_i) + \overset{\lambda}{n}{}_i^* \quad \begin{array}{l}\text{measurement/state}\\ \text{relationships}\end{array} \tag{2}$$

The respective $D\times D$ measurement error covariance matrices are denoted by $\overset{\lambda}{R}_i$. All measurement errors are assumed to be independent and Gaussian with zero mean so that the $\overset{\lambda}{R}_i$ are diagonal. When, for example, no rate measurements $\dot{r}_i^*$ are available, $\dot{R}_i^{-1}$ is taken as the null matrix which corresponds to infinite variances.

The WLS error cost function $\Lambda(\boldsymbol{p})$ to be minimized is

$$\Lambda(\boldsymbol{p}) = \sum_{i=1}^{I}\sum_{\lambda=0}^{2} (\overset{\lambda}{H}_i - \overset{\lambda}{r}{}_i^*)^T (\overset{\lambda}{R}_i)^{-1} (\overset{\lambda}{H}_i - \overset{\lambda}{r}{}_i^*) \quad \begin{array}{l}\text{cost}\\ \text{function}\end{array} \tag{3}$$

In the sequel, $\overset{\lambda}{H}_i$ is linearized through gradient terms about nominal values $r_i^0,\ldots,\overset{\lambda}{r}{}_i^0$ and $\boldsymbol{B}_i^0,\ldots,\overset{\lambda}{\boldsymbol{B}}{}_i^0$ where a zero superscript is used to denote "nominal values." When $\overset{\lambda}{H}_i$ is linearized, Λ becomes a quadratic form in $\boldsymbol{p}$.

The $\boldsymbol{B}(t)$ are parameterized in a suitable time series,

$$B^e(t) = \sum_{\alpha=1}^{M_e} \psi_\alpha^e(t) b_\alpha^e \qquad (e=1,E) \tag{4a}$$

Where ψ_α^e is a given function of t. M_e is the number of terms in the expansion. The superscript e on ψ permits different time series parameterizations (e.g., Spline, Taylor, or Fourier series) and a different number of terms M_e for each $B^e(t)$ appropriate to the manner in which the true (unknown) $B^e(t)$ is expected to vary. Because of the known powerful function representation and approximating properties of splines, a quadratic or cubic spline $\psi_\alpha^e(t)$ with variable "knot positions," T_k^e $(k=1,J^e)$ is recommended. J^e is the number of spline regions for the total time interval (t_0,t_f). We define M^E as the total number of b parameters used to represent the $E-B(t)$'s.

In matrix form, Eq. (4a) becomes

$$B(t)=\psi(t)b \qquad B(t) \text{ parameterization} \tag{4b}$$

where $\psi(t)$ is $E\times M^E$ block diagonal.

$$b^T \equiv [b_1^1 \ldots b_{M_1}^1 | \ldots | b_1^e \ldots b_\alpha^e \ldots b_{M_e}^e | \ldots | b_1^E \ldots b_{M_E}^E] \tag{4c}$$

Thus,

$$p^T=[r_0^T | \dot{r}_0^T | b^T] \qquad (2D+M^E) \text{ row vector} \tag{5}$$

where r_0, $\dot{r}_0$ are the initial conditions associated with Eq. (1).

We find that value of p given by Eq. (5) that minimizes Eq. (3) subject to the $B(t)$ parameterization Eq. (4b) and differential equation constraints Eq. (1). The result is defined as the WLS parameter estimate, $\hat{p}$. The $\hat{R}_i$ diagonal covariance matrices are updated as the arthmetic mean of the observed "residuals" squared and summed.

Method of Solution

The solution of Eq. (1) is represented accurately by the CSS

$$r(t)=r_0+\dot{r}_0(t-t_0)+\ddot{r}_0\frac{(t-t_0)^2}{2!}+\sum_{k=1}^{j-1}\psi_k(t)\,\dddot{r}_k+\frac{(t-T_j)^3}{3!}\dddot{r}_j$$

$$\psi_k(t)\equiv h_k\left[\frac{t^2}{2}-t\xi_k+\left(\frac{\xi_k^2}{2}+\frac{h_k^2}{24}\right)\right] \qquad T_j\le t\le T_{j+1} \qquad (j=1,J) \tag{6}$$

$$\dot{r}(t)=\frac{\partial}{\partial t}r(t) \qquad \ddot{r}(t)=\frac{\partial^2}{\partial t^2}r(t) \text{ and } t_0\equiv T_1 \quad t_f\equiv T_{J+1}$$

The current time t by definition lies within spline region j of time width h_j, center ξ_j, and bounds T_j, T_{j+1} $(j=1,J)$.

Equation (6) is a sequence of cubic polynomials in t with a growing number of terms $(j+3)$ as t advances. Continuity of $r(t)$, $\dot{r}(t)$, $\ddot{r}(t)$ is satisfied everywhere over (t_0,t_f). The $\dddot{r}_k$ parameters $(k=1,j)$ are constant over each spline region with a jump discontinuity occurring at the "knots" T_k, $k=2,\ldots,J$. The $\dddot{r}_k$ are evaluated at the ξ_k points $(k=1,J)$. The derivation of Eq. (6) shows that the function approximation error→0 as the $h_k\to 0$, so that in the limit, Eq. (6) becomes exact. The $\dddot{r}_k$ parameters control the "bending" of $r(t)$ in the kth spline region so that local variations of the true $r(t)$ are readily followed by choosing correct values for the $\dddot{r}_k$.

Reference 2 obtains the key result, $r(t)$, $\dot{r}(t)$, $\ddot{r}(t)$ are linear functions of p by: 1) taking d/dt of Eq. (1); 2) substituting the right-hand side (rhs) of Eq. (1) into the above expression for $\dddot{r}(t)$; 3) evaluating the rhs of $\dddot{r}(t)$ with nominal values, $r^0(t)$, $\dot{r}^0(t)$; 4) linearizing $B(t)$ about a nominal, $B^0(t)$, and substituting the rhs of Eq. (4a) for $B^e(t)$ and d/dt [Eq. (4a)] for $\dot{B}^e(t)$; 5) evaluating the rhs of $\dddot{r}(t)$ at $t=\xi_k$ $(k=1,j)$ and substituting into the CSS; and 6) linearizing the rhs of Eq. (1) evaluated at $t=t_0$ (zero) through gradient terms, then substituting for $\ddot{r}_0$ in the CSS. The result is

$$r(t)=E(t)r_0+C(t)\dot{r}_0+G(t)b+F(t) \qquad (T_j\le t\le T_{j+1}) \tag{7a}$$

$$\dot{r}(t)=\frac{\partial}{\partial t}r(t) \qquad \ddot{r}(t)=\frac{\partial^2}{\partial t^2}r(t) \qquad (j=1,J) \tag{7b}$$

Where E, C, G, F are known time-dependent matrices having dimensions $D\times D$, $D\times D$, $D\times M^E$, $D\times 1$, respectively, and consisting of simple closed-form analytical expressions. Equations (7a, b) show the linear dependence on p. Equation (7a) is a cubic spline that upon convergence of the iterations in the sequel, will satisfy the differential equation constraints Eq. (1) at the ξ_k points. Thus, numerical integration per se is not required.

Quasilinearization of Eq. (2) when substituted into Eq. (3), followed by use of Eqs. (7a,b), (4b), and $\partial\Lambda/\partial p=0$, gives $(2D+M^E)$ linear algebraic equations:

$$P(i)\hat{p}=Q(i) \qquad (i=1,I) \tag{8a}$$

$$P(i)=P(i-1)+\Delta P_{i/i-1} \qquad Q(i)=Q(i-1)+\Delta Q_{i/i-1} \tag{8b}$$

Equations (8b) show the recursive update of the P, Q matrix elements due to the current ith observation. The inversion in Eq. (8a) need only be performed for $i=I$. However, Eq. (8a) is the form obtained for all of the spline algorithms with the P,Q matrix elements specifically defined in Ref. 2. Also, Cov $(\hat{p})\cong P^{-1}$.

The dimensions of P,Q are independent of spline regions or observation times and only Q contains the observations. Equations (8) define an iterative batch algorithm for obtaining $\hat{p}$. Given $\hat{p}$ from Eq. (8a), $\overset{\lambda}{r}(t)$ $\lambda=0,1,2$ follows immediately from Eq. (7) and $B(t)$ from Eq. (4b). These quantities become the nominal values for the next iteration whereby a new set of E, C, G, F, H, R, P, Q matrices are computed, continuing in this manner until $\hat{p}$ converges. The converged values are the spline WLS parameter, $B(t)$, and $\overset{\lambda}{r}(t)$ estimates. Spline recursive nonlinear sequential parameter and state estimation for real time applications is discussed in the full paper along with other estimation topics. The latter includes a spline starting estimates algorithm (SSEA) which of itself is a complete algorithm for the smoothing or filtering of data. (The $\dddot{r}$ in Eq. (6) are treated as independent unknown parameters.) The SSEA is used to obtain the initial estimates of p, $B(t)$, $\overset{\lambda}{r}(t)$ of the SWLS algorithm. Where the measurement noise is approximately zero mean uncorrelated Guassian, experience has indicated close agreement between the two sets of results.

Conclusions

The preceding formulas specialize to first order nonlinear dynamic systems as shown in Ref. 2. Generalization to higher-order systems is straightforward. In essence, the Spline model[1] provides both a useful means of time series modeling, and a precise analytical method of converting a problem in optimal estimation or optimal control, with differential equation constraints, into a problem in algebra. (Reference 8 provides an alternate approach for optimal control.) For large complicated systems, a large reduction in computation is anticipated with the use of this technique.

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References

[1] Schneider, H. and Reddy,, P., "Spline Method for Nonlinear Optimal Thrust Vector Controls for Atmospheric Interceptor Guidance," *AIAA Journal,* Vol. 15, April 1977, pp. 449-450.

[2] Schneider, H., "Method of Time Series Modeling, Parameter and State Estimation of Nonlinear Systems" RCA Radar, Missile, and Support Systems Division, Moorestown, N.J., AD-E-SPY-T-2036, Sept. 1978.

[3] Schneider, H., MIT Lincoln Laboratory Reports, 1967-1971.

[4] Schneider, H. and Gordon, G., "Real Time Spline Methods for Nonlinear Estimation," *Second Symposium on Nonlinear Estimation Theory and Its Applications,* September 13-15, 1971, Western Periodicals Pub. Co., N. Hollywood, Calif., pp. 288-307.

[5] Schneider, H. and Ramachandran, S., "Method of Modeling and Nonlinear Parameter and State Estimation Using Splines," AAS/AIAA Astrodynamics Conference Paper, Grand Teton National Park, Wy., Sept. 7-9, 1977.

[6] Stalford, H. and Park, C., "Optimal Spline Method with Applications to Ship Classification and to Aircraft System Identification," Dynamic Research Corp., Rept. 23, March 1977.

[7] Schneider, H., "Part II. Realtime Algorithms for Sensitivity Error Analyses, Spline Estimation and Integration," Lockheed Missiles and Space Co., Tech. Memo., LMSC 19-65-HS-03, May 4, 1979. (PLSS Project).

[8] Balakrishnan, A.V., "On a New Computing Technique in Optimal Control and its Application to Minimal-Time Flight Profile Optimalization," *Journal of Optimazation Theory,* Vol. 4, July 1969, pp. 1-21.